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SOLUTION OF AN INFINITE SYSTEM OF DIFFERENTIAL EQUATIONS OF THE ANALYTIC TYPE

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Consider the infinite system of differential equations*

$$\frac{dx_i}{dt} = f_i(t; x_1, x_2, \dots) = a_i + f_i^{(1)} + f_i^{(2)} + \dots \quad (i = 1, 2, \dots), \quad (1)$$

where a_i is a constant and $f_i^{(j)}$ is the totality of terms of f_i which are homogeneous in t, x_1, x_2, \dots of degree j . That is, $f_i^{(1)}$ is a linear function of the infinitely many variables t, x_1, x_2, \dots ; $f_i^{(2)}$ is a quadratic function of the same variables; and so on. From the analogy with analytic functions of a finite number of variables, f_i will be said to be of the analytic type.

It is assumed that the following hypotheses are satisfied:

(H₁). $x_i = 0$ ($i = 1, 2, \dots$) at $t = 0$.

(H₂). *Finite real positive constants $c_0, c_1, c_2, \dots; r_0, r_1, r_2, \dots, A$ and a exist such that*

$$s = c_0 t + c_1 x_1 + c_2 x_2 + \dots \quad (2)$$

converges if

$$|t| \leq r_0, \quad |x_i| \leq r_i \quad (i = 1, 2, \dots) \quad (3)$$

and such that $Ar_i s$ dominates $f_i^{(j)}$ and $|a_i| \leq Ar_i a$.

Since the series (2) converges if the relations (3) are satisfied, a finite constant M exists such that

$$S = c_0 \frac{r_0}{M} + c_1 \frac{r_1}{M} + c_2 \frac{r_2}{M} + \dots < 1. \quad (4)$$

Consequently if $M|t| \leq r_0$ and $M|x_i| \leq r_i$ are satisfied, then

$$|f_i| \leq A r_i \{a + S + S^2 + \dots\} = A r_i \left\{ a + \frac{S}{1-S} \right\}. \quad (5)$$

* The problem of an infinite system of differential equations was first treated by E. H. Moore in his paper read at the Fourth International Congress of Mathematicians, held at Rome in 1908. His treatment was in the sense of General Analysis (cf. *Introduction to a Form of General Analysis*, New Haven Mathematical Colloquium, 1906), in which a general variable p is used in place of the index i of this paper having the range of positive integers, and his functions were not limited to those of the analytic type. The solution was made to depend upon the solution of an integral equation, in general non-linear, of the Volterra type. Simplifications and extensions of this theory were presented before the National Academy of Sciences, at Chicago, December 9, 1914.

Therefore under these limitations not only do the terms of each degree in the right member of equations (1) converge, but the whole right members converge.

The particular form of (H_2) was chosen so as to secure by one analysis as wide a range as possible of permissible values of x_1, x_2, \dots . For example, if infinitely many of the c_i are bounded from zero the x_i must tend to zero for $i = \infty$. On the other hand, if Σc_i converges all the conditions so far imposed can be satisfied by x_i which are bounded from zero. In the latter case Σf_i does not converge.

If an analytic solution of (1) satisfying the initial conditions (H_1) exists, it will have the form

$$x_i = A_i^{(1)}t + A_i^{(2)}t^2 + \dots \quad (i = 1, 2, \dots). \quad (6)$$

On substituting these series in (1) and equating coefficients of corresponding powers of t , it is found that

$$\begin{aligned} a_i^{(1)} &= a_i & (i = 1, 2, \dots), \\ 2a_i^{(2)} &= \frac{\partial f_i^{(1)}}{\partial t} + \sum_{j=1}^{\infty} \frac{\partial f_i^{(1)}}{\partial x_j} a_j^{(1)}, \\ &\dots\dots\dots, \\ na_i^{(n)} &= P_i^{(n)}(a_j^{(1)}, \dots, a_j^{(n-1)}) & (j = 1, 2, \dots), \end{aligned} \quad (7)$$

where $P_i^{(n)}$ is a polynomial in $a_j^{(1)}, \dots, a_j^{(n-1)}$ whose coefficients are linear functions of the coefficients of f_i with positive numerical multipliers. Hence the formal analytic solution of (1) is unique.

In order to prove the convergence of the series (6) for values of t whose moduli are sufficiently small, consider the solution of

$$\frac{d\xi_i}{dt} = A r_i \left\{ a + \frac{\sigma}{1 - \sigma} \right\} \quad (i = 1, 2, \dots), \quad (8)$$

where

$$\sigma = c_0 t + c_1 \xi_1 + c_2 \xi_2 + \dots \quad (9)$$

The right members of (8) dominate the respective right members of (1).

The formal analytic solution of (8) is

$$\xi_i = \alpha_i^{(1)} t + \alpha_i^{(2)} t^2 + \dots \quad (i = 1, 2, \dots). \quad (10)$$

The coefficients of these series can be obtained by equations analogous to (7). They are therefore real and positive, and it follows from the fact that the right members of (8) dominate the right members of (1) that

$$\alpha_i^{(j)} \geq |a_i^{(j)}| \quad (i, j = 1, 2, \dots). \quad (11)$$

Therefore if (10) converge for $|t| \leq \rho$, then (6) also converge for at least the same values of t .

It follows from (8) that

$$\frac{1}{r_1} \frac{d\xi_1}{dt} = \frac{1}{r_2} \frac{d\xi_2}{dt} = \dots = \frac{d\xi}{dt}.$$

The initial values of ξ_1, ξ_2, \dots are zero; hence on taking $\xi(0) = 0$, it follows that

$$\xi_i = r_i \xi \quad (i = 1, 2, \dots). \quad (12)$$

Therefore each of equations (8) reduces to

$$\frac{d\xi}{dt} = A \left\{ a + \frac{c_0 t + C\xi}{1 - c_0 t - C\xi} \right\}, \quad (13)$$

where

$$C = c_1 r_1 + c_2 r_2 + \dots, \quad (14)$$

which is a finite constant by (H_2) .

It follows from the ordinary theory for a finite number of differential equations that (13) has an analytic solution which converges if $|t|$ is sufficiently small. Therefore equations (10) and (6) converge for at least the same values of t .

In general the limitations placed on t in order that the solution of (13) shall be known to converge are so restrictive that the corresponding x_i do not attain the boundary of the region for which the right members of (1) converge. The question arises whether the solution can be continued beyond its original domain.

Suppose equations (6) converge for $t = t_0$ and let the corresponding value of x_i be $x_i^{(0)}$. Suppose

$$c_0 |t_0| + c_1 |x_1^{(0)}| + c_2 |x_2^{(0)}| + \dots = S_0 < S_1 < 1.$$

Then let

$$x_i = x_i^{(0)} + y_i, \quad t = t_0 + \tau. \quad (15)$$

The differential equations (1) become in the new variables

$$\frac{dy_i}{d\tau} = b_i + g_i^{(1)} + g_i^{(2)} + \dots \quad (i = 1, 2, \dots), \quad (16)$$

where $g_i^{(j)}$ is the totality of terms in the i th equation which are homogeneous in τ, y_1, y_2, \dots of degree j .

It follows from (1) that the explicit expressions for $b_i, g_i^{(1)}, g_i^{(2)}, \dots$ are

$$\begin{aligned} b_i &= f_i(t_0; x_1^{(0)}, x_2^{(0)}, \dots), \\ g_i^{(1)} &= \frac{\partial f_i}{\partial t} \tau + \frac{\partial f_i}{\partial x_1} y_1 + \frac{\partial f_i}{\partial x_2} y_2 + \dots, \\ &\dots \dots \dots \\ g_i^{(n)} &= \frac{1}{n!} \left[\frac{\partial f_i}{\partial t} \tau + \frac{\partial f_i}{\partial x_1} y_1 + \frac{\partial f_i}{\partial x_2} y_2 + \dots \right]^n, \\ &\dots \dots \dots \end{aligned} \quad (17)$$

where the power indicated in the last equation is symbolic such that

$$\left(\frac{\partial f_i}{\partial t} \tau \right)^{n_0} \left(\frac{\partial f_i}{\partial x_1} y_1 \right)^{n_1} \left(\frac{\partial f_i}{\partial x_2} y_2 \right)^{n_2} \dots = \frac{\partial^n f_i}{\partial t^{n_0} \partial x_1^{n_1} \partial x_2^{n_2} \dots} \tau^{n_0} y_1^{n_1} y_2^{n_2} \dots$$

$$(n_0 + n_1 + \dots = n). \quad (18)$$

After the partial derivatives have been formed, t, x_1, x_2, \dots are replaced by $t_0, x_1^{(0)}, x_2^{(0)}, \dots$ respectively.

Since the transformation (15) is linear, $g_i^{(n)}$ depends only on $f_i^{(n)}, f_i^{(n+1)}, f_i^{(n+2)}, \dots$. It follows from the fact that $f_i^{(n)}, f_i^{(n+1)}, \dots$ are dominated by $Ar_i S^n, Ar_i S^{n+1}, \dots$ respectively that

$$\left| \frac{\partial^n f_i}{\partial t^{n_0} \partial x_1^{n_1} \partial x_2^{n_2} \dots} \right|_{\substack{t=|t_0| \\ x_j=|x_j^{(0)}|}} \leq Ar_i \left[\frac{\partial^n \left(\frac{S^n}{1-S} \right)}{\partial t^{n_0} \partial x_1^{n_1} \partial x_2^{n_2} \dots} \right]_{\substack{t=|t_0| \\ x_j=|x_j^{(0)}|}}.$$

It is easily found that

$$\left[\frac{\partial^n \left(\frac{S^n}{1-S} \right)}{\partial t^{n_0} \partial x_1^{n_1} \partial x_2^{n_2} \dots} \right]_{\substack{t=|t_0| \\ x_j=|x_j^{(0)}|}} = \frac{n! c_0^{n_0} c_1^{n_1} c_2^{n_2} \dots}{(1-S_0)^{n+1}}. \quad (19)$$

Now let

$$c_0^{(1)} = \frac{c_0}{1-S_0}, \quad c_1^{(1)} = \frac{c_1}{1-S_0}, \quad c_2^{(1)} = \frac{c_2}{1-S_0} \quad (20)$$

The series

$$T = c_0^{(1)} \tau + c_1^{(1)} y_1 + c_2^{(1)} y_2 + \dots$$

converges if

$$|\tau| \leq r_0 (1-S_0), \quad |y_1| \leq r_1 (1-S_0), \quad |y_2| \leq r_2 (1-S_0), \quad \dots,$$

and $|T| < 1$ if

$$|\tau| \leq \frac{r_0 (1-S_0)}{M}, \quad |y_i| \leq \frac{r_i (1-S_0)}{M}.$$

Moreover $g_i^{(n)}$ is dominated by $\frac{Ar_i}{1-S_0} T^n$. Therefore equations (16) have the essential properties assumed to hold for (1), and their solution converges for $|\tau|$ sufficiently small.

It is easy to imagine a physical problem which satisfies the conditions of this theory. For example, suppose the number of mutually gravitating bodies in the universe whose masses are bounded from zero is infinite. If beyond a finite number of them (which may be arbitrarily great) their initial distances from one another increase, as the number of bodies increases, with sufficient rapidity, it is easy to show that all the hypotheses are satisfied. In this case there is a rigorous, though limited, solution of the problem of infinitely many bodies moving subject to their mutual attractions.

SEX RATIO IN PIGEONS, TOGETHER WITH OBSERVATIONS ON THE LAYING, INCUBATION AND HATCHING OF THE EGGS

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The conclusions¹ here presented are the results of a study of the records which have accumulated from 1907 to 1914 in connection with investigations of inheritance in pigeons begun at the Rhode Island Agricultural Experiment Station and later continued at the Experiment Station at Madison, Wisconsin. It is impossible in a brief space to present the data upon which the conclusions are based; for these the reader is referred to the complete report. Furthermore, although the conclusions are here presented somewhat dogmatically and as if of general application, and while we believe that they will probably be found in the main to apply generally to domestic pigeons, they are nevertheless based almost entirely on the data of the experiments mentioned and there is, therefore, no positive assurance that the results would be the same with other stock or under different conditions. The number of data obtained were, however, very considerable for pigeons, and it is felt they accordingly furnish a good foundation for the conclusions drawn.

It is commonly believed by pigeon breeders and others that from the two eggs of a clutch a pair of offspring, that is a male and a female, are produced either invariably, or at least in a great majority of instances. Furthermore, it is maintained that of this pair the male hatches from the egg which is laid first, while the egg laid later produces the female. The